

$N_6$  PROPERTY FOR THIRD VERONESE EMBEDDINGS

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ABSTRACT. The rational homology groups of the matching complexes are closely related to the syzygies of the Veronese embeddings. In this paper we will prove the vanishing of certain rational homology groups of matching complexes, thus proving that the third Veronese embeddings satisfy the property  $N_6$ . This settles the Ottaviani-Paoletti conjecture for third Veronese embeddings. This result is optimal since  $\nu_3(\mathbb{P}^n)$  does not satisfy the property  $N_7$  for  $n \geq 2$  as shown by Ottaviani-Paoletti in [OP].

## 1. INTRODUCTION

Let  $k$  be a field of characteristic 0. Let  $V$  be a finite dimensional vector space over  $k$  of dimension  $n + 1$ . The projective space  $\mathbb{P}(V)$  has coordinate ring naturally isomorphic to  $\text{Sym } V$ . For each natural number  $d$ , the  $d$ -th Veronese embedding of  $\mathbb{P}(V)$ , which is naturally embedded into the projective space  $\mathbb{P}(\text{Sym}^d V)$  has coordinate ring  $\text{Ver}(V, d) = \bigoplus_{k=0}^{\infty} \text{Sym}^{kd} V$ . For each set of integers  $p, q, b$ , let  $K_{p,q}^d(V, b)$  be the associated Koszul cohomology group defined as the homology of the 3 term complex

$$\begin{aligned} \bigwedge^{p+1} \text{Sym}^d V \otimes \text{Sym}^{(q-1)d+b} V &\rightarrow \bigwedge^p \text{Sym}^d V \otimes \text{Sym}^{qd+b} V \\ &\rightarrow \bigwedge^{p-1} \text{Sym}^d V \otimes \text{Sym}^{(q-1)d+b} V. \end{aligned}$$

Then  $K_{p,q}^d(V, b)$  is the space of minimal  $p$ -th syzygies of degree  $p + q$  of the  $\text{GL}(V)$  module  $\bigoplus_{k=0}^{\infty} \text{Sym}^{kd+b} V$ . We write  $K_{p,q}^d(b) : \text{Vect} \rightarrow \text{Vect}$  for the functor on finite dimensional  $k$ -vector spaces that assigns to a vector space  $V$  the corresponding syzygy module  $K_{p,q}^d(V, b)$ .

**Conjecture 1** (Ottaviani-Paoletti).

$$K_{p,q}^d(V, 0) = 0 \text{ for } q \geq 2 \text{ and } p \leq 3d - 3.$$

The conjecture is known for  $d = 2$  by the work of Josefiak-Pragacz-Weyman [JPW] and also known for  $\dim V = 2$  and  $\dim V = 3$  by the work of Green [G] and Birkenhake [B]. All other cases are open. Also, Ottaviani and Paoletti in [OP] showed that  $K_{p,2}^d(V, 0) \neq 0$  for  $p = 3d - 2$ , when  $\dim V \geq 3$ ,  $d \geq 3$ . In other words, the conjecture is sharp. Recently, Bruns, Conca, and Römer in [BCR] showed that  $K_{d+1,q}^d(V, 0) = 0$  for  $q \geq 2$ . In this paper we will prove the conjecture in the case  $d = 3$ . Thus the main theorem of the paper is:

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**Theorem 3.6.** The third Veronese embeddings of projective spaces satisfy property  $N_6$ .

To prove Theorem 3.6 we prove that vanishing results hold for certain rational homology groups of matching complexes (defined below), and then use [KRW, Theorem 5.3] to translate between the zyzygy modules  $K_{p,q}^d(V, b)$  and the homology groups of matching complexes.

**Definition 2** (Matching Complexes). *Let  $d > 1$  be a positive integer and  $A$  a finite set. The matching complex  $C_A^d$  is the simplicial complex whose vertices are all the  $d$ -element subsets of  $A$  and whose faces are  $\{A_1, \dots, A_r\}$  so that  $A_1, \dots, A_r$  are mutually disjoint.*

The symmetric group  $S_A$  acts on  $C_A^d$  by permuting the elements of  $A$  making the homology groups of  $C_A^d$  representations of  $S_A$ . For each partition  $\lambda$ , we denote by  $V^\lambda$  the irreducible representation of  $S_{|\lambda|}$  corresponding to the partition  $\lambda$ , and  $S_\lambda$  the Schur functor corresponding to the partition  $\lambda$ . For each vector space  $V$ ,  $S_\lambda(V)$  is an irreducible representation of  $\text{GL}(V)$ . The relation between the syzygies of the Veronese embeddings and the homologies of matching complexes is given by the following theorem of Karaguezian, Reiner and Wachs.

**Theorem 1.1.** [KRW, Theorem 5.3] *Let  $p, q$  be non-negative integers, let  $d$  be a positive integer and let  $b$  be a non-negative integer. Write  $N = (p + q)d + b$ . Consider a partition  $\lambda$  of  $N$ . Then the multiplicity of  $S_\lambda$  in  $K_{p,q}^d(b)$  coincides with the multiplicity of the irreducible  $S_N$  representation  $V^\lambda$  in  $\tilde{H}_{p-1}(C_N^d)$ .*

This correspondence makes Conjecture 1 equivalent to the following conjecture.

**Conjecture 3.** *The only non-zero homology groups of  $C_{nd}^d$  for  $n = 1, \dots, 3d - 1$  is  $\tilde{H}_{n-2}$ .*

We will prove this conjecture for  $d = 3$  by computing the homology groups of  $C_n^3$  by induction. To compute the homology groups of the matching complexes inductively, the following equivariant long-exact sequence introduced by Raicu in [R] is useful. Let  $A$  be a finite set with  $|A| \geq 2d$ . Let  $a \in A$  be an element of  $A$ . Let  $\alpha$  be a  $d$ -element subset of  $A$  such that  $a \in \alpha$ . Let  $\beta = \alpha \setminus a$ , and let  $C = A \setminus \alpha$ ,  $B = A \setminus \{a\}$ . Then we have the following long-exact sequence of representations of  $S_B$ .

$$(1.1) \quad \begin{aligned} \cdots \rightarrow \text{Ind}_{S_C \times S_\beta}^{S_B}(\tilde{H}_r(C_C^d) \otimes 1) &\rightarrow \tilde{H}_r(C_B^d) \rightarrow \text{Res}_{S_A}^{S_B}(\tilde{H}_r(C_A^d)) \\ &\rightarrow \text{Ind}_{S_C \times S_\beta}^{S_B}(\tilde{H}_{r-1}(C_C^d) \otimes 1) \rightarrow \cdots \end{aligned}$$

Moreover, for each  $b$ ,  $0 \leq b \leq d - 1$ , the  $\text{GL}(V)$  module  $\bigoplus_{k=0}^{\infty} \text{Sym}^{kd+b} V$  is a Cohen-Macaulay module in the coordinate ring of the projective space  $\mathbb{P}(\text{Sym}^d V)$ . The dual of the resolution of each of these modules is the resolution of another such module giving us the duality among the Koszul homology groups  $K_{p,q}^d(V, b)$ . For simplicity, since we deal with third Veronese embeddings, we will from now on assume that  $d = 3$ . To compute the homology groups of the matching complexes  $C_n^3$  for  $n \leq 10$  we use the duality in the case  $\dim V = 2$ , and to compute the homology groups of the matching complexes  $C_n^3$  for  $n \geq 14$ , we use the duality in the case  $\dim V = 3$ , so we will make them explicit here. In the case  $\dim V = 2$ , the

canonical module of  $\text{Ver}(V, 3)$  as representation of  $\text{GL}(V)$  is  $S_\lambda(V)$  with  $\lambda = (5, 5)$ . From [W, Chapter 2],

$$\text{Hom}(S_{(a,b)}, S_{(5,5)}) \cong S_{(5-b, 5-a)}$$

and the correspondence in Theorem 1.1, we have

**Proposition 1.2.** *The multiplicity of  $V^\lambda$  with  $\lambda = (\lambda_1, \lambda_2)$  in the homology group  $\tilde{H}_{p-1}(C_N^3)$  coincides with the multiplicity of  $V^\mu$  with  $\mu = (5 - \lambda_2, 5 - \lambda_1)$  in the homology group  $\tilde{H}_{1-p}(C_{10-N}^3)$ .*

Similarly, in the case  $\dim V = 3$ , the canonical module of  $\text{Ver}(V, 3)$  as representation of  $\text{GL}(V)$  is  $S_\lambda(V)$  with  $\lambda = (9, 9, 9)$ . From [W, Chapter 2],

$$\text{Hom}(S_{(a,b,c)}, S_{(9,9,9)}) \cong S_{(9-c, 9-b, 9-a)}$$

and the correspondence in Theorem 1.1, we have

**Proposition 1.3.** *The multiplicity of  $V^\lambda$  with  $\lambda = (\lambda_1, \lambda_2, \lambda_3)$  in the homology group  $\tilde{H}_{p-1}(C_N^3)$  coincides with the multiplicity of  $V^\mu$  with  $\mu = (9 - \lambda_3, 9 - \lambda_2, 9 - \lambda_1)$  in the homology group  $\tilde{H}_{6-p}(C_{27-N}^3)$ .*

Finally, to determine the homology groups of the matching complex  $C_N^3$ , we apply the equivariant long exact sequence (1.1) to derive equalities and inequalities for the multiplicities of the possible irreducible representations in the unknown homology groups of  $C_N^3$ . This is carried out with the help of our Macaulay2 package MatchingComplex.m2 that we will explain in the appendix. We then use Maple to solve this system of equalities and inequalities.

The paper is organized as follows. In the second section, using the equivariant long exact sequence (1.1) and the duality in Proposition 1.2 we compute homology groups of matching complexes  $C_n^3$  for  $n \leq 13$ . In the third section, using the results in the second section and the duality in Proposition 1.3 we derive all irreducible representations whose corresponding partitions have at most 3 rows in the homology groups of  $C_n^3$  for  $n \geq 14$ . The systems of equalities and inequalities from exact sequences obtained by applying the equivariant long exact sequence (1.1) with  $|A| = 20$  and  $|A| = 23$  will not determine  $\tilde{H}_4(C_{20}^3)$  and  $\tilde{H}_5(C_{23}^3)$  uniquely. Thus we need to compute the dimensions of  $K_{p,1}(V, 2)$  with  $p = 5, 6$  and  $\dim V = 4$ . This is done by using Macaulay2 to compute the dimensions of the spaces of minimal  $p$ -th syzygies of degree  $p+1$  for  $p = 6, 7$  of the module  $\bigoplus_{k=0}^{\infty} \text{Sym}^{3k+2} V$  with  $\dim V = 4$ . We then state the results for the homology groups of  $C_n^3$  with  $14 \leq n \leq 24$  and finish the proof of our main theorem. The proof of Proposition 3.5 illustrates the computation of the homology groups of the matching complexes  $C_n^3$  dealing with the most complicated matching complex  $C_{23}^3$  in the series. In the appendix we explain the ideas behind our package leading to the computation of the homology groups of the matching complexes.

## 2. HOMOLOGY OF MATCHING COMPLEXES

In this section, using the equivariant long exact sequence (1.1) we compute the homology groups of the matching complexes  $C_n^3$  for  $n \leq 13$ . In the following, we denote the partition  $\lambda$  with row lengths  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k \geq 0$  by the sequence  $(\lambda_1, \lambda_2, \dots, \lambda_k)$  and we use the same notation for the representation  $V^\lambda$ . To simplify notation, we omit the subscript and superscript when we use the operators  $\text{Ind}$  and

Res. It is clear from the context and the equivariant long exact sequence what the induction and restriction are. From the definition of the matching complexes, it is not hard to see the following.

**Proposition 2.1.** *The only non-vanishing homology groups of  $C_n^3$  with  $n = 4, 5, 6$  are respectively*

$$\tilde{H}_0 C_4^3 = (3, 1), \quad \tilde{H}_0 C_5^3 = (4, 1) \oplus (3, 2), \quad \tilde{H}_0 C_6^3 = (4, 2).$$

Together with Proposition 1.2 we have

**Proposition 2.2.** *The only irreducible representations whose corresponding partitions have at most 2 rows in the homology groups of the matching complexes  $C_n^3$ ,  $7 \leq n \leq 10$  are*

$$(\tilde{H}_1 C_8^3)_2 = (5, 3), \quad (\tilde{H}_1 C_9^3)_2 = (5, 4), \quad (\tilde{H}_1 C_{10}^3)_2 = (5, 5)$$

where  $(\tilde{H}_i C_n^3)_2$  denote the subrepresentation of  $\tilde{H}_i C_n^3$  consisting of all irreducible representations of  $\tilde{H}_i C_n^3$  whose corresponding partitions have at most 2 rows.

**Proposition 2.3.** *The only non-vanishing homology group of  $C_7^3$  is*

$$\tilde{H}_1 C_7^3 = (5, 1, 1) \oplus (3, 3, 1).$$

*Proof.* Applying the equivariant long exact sequence (1.1) with  $|A| = 7$  we have exact sequences

$$0 \rightarrow \text{Res } \tilde{H}_i C_7^3 \rightarrow 0$$

for  $i \neq 0, 1$ , and an exact sequence

$$0 \rightarrow \text{Res } \tilde{H}_1 C_7^3 \rightarrow \text{Ind } \tilde{H}_0 C_4^3 \rightarrow \tilde{H}_0 C_6^3 \rightarrow \text{Res } \tilde{H}_0 C_7^3 \rightarrow 0.$$

Therefore,  $\tilde{H}_i C_7^3 = 0$  for  $i \neq 0, 1$ . To show that  $\tilde{H}_0 C_7^3$  is zero, note that  $\tilde{H}_0 C_6^3$  maps surjectively onto  $\text{Res } \tilde{H}_0 C_7^3$ . Since  $\tilde{H}_0 C_6^3 = (4, 2)$  and by Proposition 2.2,  $\tilde{H}_0 C_7^3$  does not contain any irreducible representations whose corresponding partitions have at most 2 rows, it must be zero. Therefore, we know that  $\text{Res } \tilde{H}_1 C_7^3$  as representation of  $S_6$  is equal to

$$\text{Ind } \tilde{H}_0 C_4^3 - \tilde{H}_0 C_6^3 = (5, 1) \oplus (4, 1, 1) \oplus (3, 3) \oplus (3, 2, 1).$$

Moreover, by Proposition 2.2,  $\tilde{H}_1 C_7^3$  does not contain any irreducible representations whose corresponding partitions have at most 2 rows, thus  $\tilde{H}_1 C_7^3$  must consist  $(5, 1, 1)$  and  $(3, 3, 1)$  as its restriction contains  $(5, 1)$  and  $(3, 3)$ . But the restriction of  $(5, 1, 1) \oplus (3, 3, 1)$  is equal to  $\text{Res } \tilde{H}_1 C_7^3$  so we have the desired conclusion.  $\square$

**Proposition 2.4.** *The only non-vanishing homology group of  $C_8^3$  is*

$$\tilde{H}_1 C_8^3 = (6, 1, 1) \oplus (5, 2, 1) \oplus (5, 3) \oplus (4, 3, 1) \oplus (3, 3, 2).$$

*Proof.* Applying the equivariant long exact sequence (1.1) with  $|A| = 8$  we have exact sequences

$$0 \rightarrow \text{Res } \tilde{H}_i C_8^3 \rightarrow 0$$

for  $i \neq 1$ , and an exact sequence

$$0 \rightarrow \tilde{H}_1 C_7^3 \rightarrow \text{Res } \tilde{H}_1 C_8^3 \rightarrow \text{Ind } \tilde{H}_0 C_5^3 \rightarrow 0.$$

Therefore,  $\tilde{H}_i C_8^3 = 0$  for  $i \neq 1$ . To compute  $\tilde{H}_1 C_8^3$ , note that from the exact sequence

$$\text{Res } \tilde{H}_1 C_8^3 = \tilde{H}_1 C_7^3 + \text{Ind } \tilde{H}_0 C_5^3$$

which consists of irreducible representations whose corresponding partitions have at most 3 rows, therefore  $\tilde{H}_1 C_8^3$  contains only irreducible representations whose corresponding partitions have at most 3 rows. Moreover, by Proposition 2.2, the only irreducible representation whose corresponding partition has at most 2 rows in  $\tilde{H}_1 C_8^3$  is  $(5, 3)$ . Therefore, the restrictions of irreducible representations whose corresponding partitions have 3 rows in  $\tilde{H}_1 C_8^3$  is equal to

$$(3, 2, 2) \oplus 2 \cdot (3, 3, 1) \oplus 2 \cdot (4, 2, 1) \oplus (4, 3) \oplus 2 \cdot (5, 1, 1) \oplus (5, 2) \oplus (6, 1).$$

It is easy to see that the set of irreducible representations whose corresponding partitions have 3 rows in  $\tilde{H}_1 C_8^3$  has to be equal to  $(6, 1, 1) \oplus (5, 2, 1) \oplus (4, 3, 1) \oplus (3, 3, 2)$ .  $\square$

**Proposition 2.5.** *The only non-vanishing homology group of  $C_9^3$  is*

$$\tilde{H}_1 C_9^3 = (6, 2, 1) \oplus (5, 4) \oplus (5, 3, 1) \oplus (4, 3, 2).$$

*Proof.* Applying the equivariant long exact sequence (1.1) with  $|A| = 9$  we have exact sequences

$$0 \rightarrow \text{Res } \tilde{H}_i C_9^3 \rightarrow 0$$

for  $i \neq 1$ , and an exact sequence

$$0 \rightarrow \tilde{H}_1 C_8^3 \rightarrow \text{Res } \tilde{H}_1 C_9^3 \rightarrow \text{Ind } \tilde{H}_0 C_6^3 \rightarrow 0.$$

Therefore,  $\tilde{H}_i C_9^3 = 0$  for  $i \neq 1$ , and

$$\text{Res } \tilde{H}_1 C_9^3 = \tilde{H}_1 C_8^3 + \text{Ind } \tilde{H}_0 C_6^3.$$

Moreover, by Proposition 2.2, the only irreducible representation whose corresponding partition has at most 2 rows in  $\tilde{H}_1 C_9^3$  is  $(5, 4)$ . Therefore, the restrictions of irreducible representations whose corresponding partitions have 3 rows in  $\tilde{H}_1 C_9^3$  is equal to

$$(3, 3, 2) \oplus (4, 2, 2) \oplus 2 \cdot (4, 3, 1) \oplus 2 \cdot (5, 2, 1) \oplus (5, 3) \oplus (6, 1, 1) \oplus (6, 2).$$

It is easy to see that the set of irreducible representations whose corresponding partitions have 3 rows in  $\tilde{H}_1 C_9^3$  has to be equal to  $(6, 2, 1) \oplus (5, 3, 1) \oplus (4, 3, 2)$ .  $\square$

**Proposition 2.6.** *The only non-vanishing homology groups of  $C_{10}^3$  are*

$$\tilde{H}_2 C_{10}^3 = (7, 1, 1, 1) \oplus (5, 3, 1, 1) \oplus (3, 3, 3, 1), \quad \tilde{H}_1 C_{10}^3 = (5, 5).$$

*Proof.* Applying the equivariant long exact sequence (1.1) with  $|A| = 10$  we have exact sequences

$$0 \rightarrow \text{Res } \tilde{H}_i C_{10}^3 \rightarrow 0$$

for  $i \neq 1, 2$ , and an exact sequence

$$0 \rightarrow \text{Res } \tilde{H}_2 C_{10}^3 \rightarrow \text{Ind } \tilde{H}_1 C_7^3 \rightarrow \tilde{H}_1 C_9^3 \rightarrow \text{Res } \tilde{H}_1 C_{10}^3 \rightarrow 0.$$

Therefore,  $\tilde{H}_i C_{10}^3 = 0$  for  $i \neq 1, 2$ . Since  $\tilde{H}_1 C_9^3$  maps surjectively onto  $\text{Res } \tilde{H}_1 C_{10}^3$ , and by Proposition 2.2,  $\tilde{H}_1 C_{10}^3$  contains the irreducible representation  $(5, 5)$ ,  $(6, 2, 1) \oplus (5, 3, 1) \oplus (4, 3, 2)$  maps surjectively onto the restrictions of the other irreducible representations of  $\tilde{H}_1 C_{10}^3$ . Moreover, there is no irreducible representation of  $S_{10}$  whose

restriction is a subrepresentation of  $(6, 2, 1) \oplus (5, 3, 1) \oplus (4, 3, 2)$ , thus  $\tilde{H}_1 C_{10}^3 = (5, 5)$ . Therefore,  $\text{Res } \tilde{H}_2 C_{10}^3$  is equal to

$$\begin{aligned} \text{Ind } \tilde{H}_1 C_7^3 + \text{Res } \tilde{H}_1 C_{10}^3 - \tilde{H}_1 C_9^3 &= (3, 3, 2, 1) \oplus (3, 3, 3) \oplus (4, 3, 1, 1) \\ &\oplus (5, 2, 1, 1) \oplus (5, 3, 1) \oplus (6, 1, 1, 1) \oplus (7, 1, 1). \end{aligned}$$

Moreover, since  $\tilde{H}_1 C_{10}^3 = (5, 5)$ , by Theorem 1.1,  $K_{2,1}(V, 1) = S_\lambda(V)$  with  $\lambda = (5, 5)$ . Since  $\bigoplus_{k=0}^{\infty} \text{Sym}^{3k+1}(V)$  with  $\dim V = 3$  is a Cohen-Macaulay module of codimension 7 with  $h$ -vector  $(3, 6)$ ,

$$\dim K_{3,0}(V, 1) = 6 \cdot \binom{7}{2} + \dim K_{2,1}(V, 1) - 3 \cdot \binom{7}{3} = 0.$$

By Theorem 1.1,  $\tilde{H}_2 C_{10}^3$  does not contain any irreducible representations whose corresponding partitions have at most 3 rows, therefore it must contain irreducible representations  $(3, 3, 3, 1)$ ,  $(5, 3, 1, 1)$ ,  $(7, 1, 1, 1)$  each with multiplicity 1. But the sum of the restrictions of these irreducible representations is equal to the restriction of  $\tilde{H}_2 C_{10}^3$  already, thus we have the desired conclusion.  $\square$

**Proposition 2.7.** *The only non-vanishing homology group of  $C_{11}^3$  is*

$$\begin{aligned} \tilde{H}_2 C_{11}^3 &= (7, 3, 1) \oplus (6, 4, 1) \oplus (6, 3, 2) \oplus (5, 4, 2) \oplus (5, 3, 3) \\ &\oplus (6, 3, 1, 1) \oplus (8, 1, 1, 1) \oplus (7, 2, 1, 1) \oplus (5, 4, 1, 1) \\ &\oplus (5, 3, 2, 1) \oplus (4, 3, 3, 1) \oplus (3, 3, 3, 2). \end{aligned}$$

*Proof.* Applying the equivariant long exact sequence (1.1) with  $|A| = 11$  we have exact sequences

$$0 \rightarrow \text{Res } \tilde{H}_i C_{11}^3 \rightarrow 0$$

for  $i \neq 1, 2$  and an exact sequence

$$0 \rightarrow \tilde{H}_2 C_{10}^3 \rightarrow \text{Res } \tilde{H}_2 C_{11}^3 \rightarrow \text{Ind } \tilde{H}_1 C_8^3 \rightarrow \tilde{H}_1 C_{10}^3 \rightarrow \text{Res } \tilde{H}_1 C_{11}^3 \rightarrow 0.$$

Therefore,  $\tilde{H}_i C_{11}^3 = 0$  for  $i \neq 1, 2$ . To show that  $\tilde{H}_1 C_{11}^3$  is zero, note that  $\tilde{H}_1 C_{10}^3$ , consists of partition  $(5, 5)$  only, maps surjectively onto  $\text{Res } \tilde{H}_1 C_{11}^3$ , so  $\tilde{H}_1 C_{11}^3 = 0$ . Thus

$$\text{Res } \tilde{H}_2 C_{11}^3 = \tilde{H}_2 C_{10}^3 + \text{Ind } \tilde{H}_1 C_8^3 - \tilde{H}_1 C_{10}^3.$$

Moreover,  $\tilde{H}_2 C_{11}^3$  does not contain any irreducible representations whose corresponding partitions have at most 2 rows, but  $\text{Res } \tilde{H}_2 C_{11}^3$  contains  $(6, 4)$  and  $(7, 3)$ , thus  $\tilde{H}_2 C_{11}^3$  must contain  $(6, 4, 1)$  and  $(7, 3, 1)$  each with multiplicity 1. The restrictions of the remaining irreducible representations is equal to

$$A = \tilde{H}_2 C_{10}^3 + \text{Ind } \tilde{H}_1 C_8^3 - \tilde{H}_1 C_{10}^3 - \text{Res}(6, 4, 1) - \text{Res}(7, 3, 1).$$

To find a set of irreducible representations whose sum of the restrictions is  $A$ , we first induce  $A$ . We then eliminate all irreducible representations whose restrictions are not contained in  $A$ . After that, we have a set of partitions  $B$ . Then we write down the set of equations that make the restriction of  $B$  equal to  $A$ . Then we solve for the non-negative integer solutions of that system. For this problem, it is easy to see that we have a unique solution as in the statement of the proposition.  $\square$

**Proposition 2.8.** *The only non-vanishing homology group of  $C_{12}^3$  is*

$$\begin{aligned}\tilde{H}_2 C_{12}^3 = & (7, 4, 1) \oplus (7, 3, 2) \oplus (6, 5, 1) \oplus (6, 4, 2) \oplus (6, 3, 3) \oplus (5, 5, 2) \\ & \oplus (5, 4, 3) \oplus (8, 2, 1, 1) \oplus (7, 3, 1, 1) \oplus (6, 4, 1, 1) \oplus (6, 3, 2, 1) \\ & \oplus (5, 4, 2, 1) \oplus (5, 3, 3, 1) \oplus (4, 3, 3, 2).\end{aligned}$$

*Proof.* Applying the equivariant long exact sequence (1.1) with  $|A| = 12$  we have exact sequences

$$0 \rightarrow \text{Res } \tilde{H}_i C_{12}^3 \rightarrow 0$$

for  $i \neq 2$  and an exact sequence

$$0 \rightarrow \tilde{H}_2 C_{11}^3 \rightarrow \text{Res } \tilde{H}_2 C_{12}^3 \rightarrow \text{Ind } \tilde{H}_1 C_9^3 \rightarrow 0.$$

Therefore,  $\tilde{H}_i C_{12}^3 = 0$  for  $i \neq 2$  and

$$\text{Res } \tilde{H}_2 C_{12}^3 = \tilde{H}_2 C_{11}^3 + \text{Ind } \tilde{H}_1 C_9^3.$$

Moreover,  $\tilde{H}_2 C_{12}^3$  does not contain any irreducible representations whose corresponding partitions have at most 2 rows, but  $\text{Res } \tilde{H}_2 C_{12}^3$  contains  $(6, 5)$  and  $(7, 4)$ , thus  $\tilde{H}_2 C_{12}^3$  must contain  $(6, 5, 1)$  and  $(7, 4, 1)$  each with multiplicity 1. Therefore, the sum of the restrictions of other irreducible representations in  $\tilde{H}_2 C_{12}^3$  is equal to

$$A = \tilde{H}_2 C_{11}^3 + \text{Ind } \tilde{H}_1 C_9^3 - \text{Res}(6, 5, 1) - \text{Res}(7, 4, 1).$$

Let  $B$  be the set containing all irreducible representations that appears in the induction of  $A$  whose restrictions are contained in  $A$ . Write down the equations that make the restriction of  $B$  equal to  $A$ . Then we solve for the non-negative integer solutions of that system. For this problem, it is easy to see that we have a unique solution as in the statement of the proposition.  $\square$

**Proposition 2.9.** *The only non-vanishing homology groups of  $C_{13}^3$  are*

$$\begin{aligned}\tilde{H}_3 C_{13}^3 = & (9, 1, 1, 1, 1) \oplus (7, 3, 1, 1, 1) \oplus (5, 5, 1, 1, 1) \oplus (5, 3, 3, 1, 1) \\ & \oplus (3, 3, 3, 3, 1) \\ \tilde{H}_2 C_{13}^3 = & (7, 5, 1) \oplus (7, 3, 3) \oplus (6, 5, 2) \oplus (5, 5, 3).\end{aligned}$$

*Proof.* Applying the equivariant long exact sequence (1.1) with  $|A| = 13$  we have exact sequences

$$0 \rightarrow \text{Res } \tilde{H}_i C_{13}^3 \rightarrow 0$$

for  $i \neq 2, 3$  and an exact sequence

$$0 \rightarrow \text{Res } \tilde{H}_3 C_{13}^3 \rightarrow \text{Ind } \tilde{H}_2 C_{10}^3 \rightarrow \tilde{H}_2 C_{12}^3 \rightarrow \text{Res } \tilde{H}_2 C_{13}^3 \rightarrow \text{Ind } \tilde{H}_1 C_{10}^3 \rightarrow 0.$$

Therefore,  $\tilde{H}_i C_{13}^3 = 0$  for  $i \neq 2, 3$ . From the sequence,  $\text{Res } \tilde{H}_3 C_{13}^3$  is contained in  $\text{Ind } \tilde{H}_2 C_{10}^3$  and contains  $\text{Ind } \tilde{H}_2 C_{10}^3 - \tilde{H}_2 C_{12}^3$ . There is a unique solution as indicated in the proposition. Therefore,

$$\text{Res } \tilde{H}_2 C_{13}^3 = \text{Res } \tilde{H}_3 C_{13}^3 - \text{Ind } \tilde{H}_2 C_{10}^3 + \tilde{H}_2 C_{12}^3 + \text{Ind } \tilde{H}_1 C_{10}^3.$$

Moreover,  $\text{Res } \tilde{H}_2 C_{13}^3$  does not contain any irreducible representations whose corresponding partitions have at most two rows, but its restriction contains  $(7, 5)$ , thus it must contain  $(7, 5, 1)$  with multiplicity 1. Therefore, the sum of the restrictions of the other irreducible representations in  $\tilde{H}_2 C_{13}^3$  is equal to

$$(5, 4, 3) \oplus 2 \cdot (5, 5, 2) \oplus (6, 3, 3) \oplus (6, 4, 2) \oplus (6, 5, 1) \oplus (7, 3, 2).$$

It is easy to see that there is a unique solution as stated in the proposition.  $\square$

## 3. PROOF OF THE MAIN THEOREM

In this section we determine the homology groups of the matching complexes  $C_n^3$  for  $14 \leq n \leq 24$  using the equivariant long exact sequence (1.1) and the duality as stated in Proposition 1.3. Note that to determine the homology groups of  $C_{20}^3$  and  $C_{23}^3$  we need to compute the dimensions of  $K_{p,0}(V, 2)$  for  $p = 6, 7$ . In the following, for a representation  $W$  of the symmetric group  $S_N$  and a positive number  $k$ , we denote by  $W_k$  the direct sum of all irreducible representations of  $W$  whose corresponding partitions have at most  $r$  rows and let  $W^k = W - W_k$ .

**Proposition 3.1.** *The only homology groups of  $C_n^3$  for  $14 \leq n \leq 24$  containing irreducible representations whose corresponding partitions have at most 3 rows are  $\tilde{H}_3 C_{14}^3, \tilde{H}_3 C_{15}^3, \tilde{H}_3 C_{16}^3, \tilde{H}_4 C_{17}^3, \tilde{H}_4 C_{18}^3, \tilde{H}_4 C_{19}^3, \tilde{H}_4 C_{20}^3, \tilde{H}_5 C_{21}^3, \tilde{H}_5 C_{22}^3$  and  $\tilde{H}_5 C_{23}^3$ . Moreover,*

$$(\tilde{H}_4 C_{20}^3)_3 = (8, 8, 4) \oplus (8, 6, 6), \text{ and } (\tilde{H}_5 C_{23}^3)_3 = (9, 8, 6).$$

*Proof.* This follows from the results of the homology of matching complexes  $C_n^3$  for  $n \leq 13$  in section 2 and the duality in Proposition 1.3.  $\square$

**Proposition 3.2.** *The only non-vanishing homology groups of  $C_n^3$  for  $14 \leq n \leq 19$  are  $\tilde{H}_3 C_{14}^3, \tilde{H}_3 C_{15}^3, \tilde{H}_3 C_{16}^3, \tilde{H}_4 C_{16}^3, \tilde{H}_4 C_{17}^3, \tilde{H}_4 C_{18}^3, \tilde{H}_4 C_{19}^3, \tilde{H}_5 C_{19}^3$ .*

*Proof.* The computational proof is given in our Macaulay2 package MatchingComplex.m2.  $\square$

**Proposition 3.3.** *The only non-vanishing homology groups of  $C_{20}^3$  are  $\tilde{H}_5 C_{20}^3$  and*

$$\tilde{H}_4 C_{20}^3 = (8, 8, 4) \oplus (8, 6, 6).$$

*Proof.* Applying the equivariant long exact sequence (1.1) with  $|A| = 20$  we have exact sequences

$$0 \rightarrow \text{Res } \tilde{H}_i C_{20}^3 \rightarrow 0$$

for  $i \neq 4, 5$  and an exact sequence

$$0 \rightarrow \tilde{H}_5 C_{19}^3 \rightarrow \text{Res } \tilde{H}_5 C_{20}^3 \rightarrow \text{Ind } \tilde{H}_4 C_{17}^3 \rightarrow \tilde{H}_4 C_{19}^3 \rightarrow \text{Res } \tilde{H}_4 C_{20}^3 \rightarrow 0.$$

Therefore,  $\tilde{H}_i C_{20}^3 = 0$  for  $i \neq 4, 5$ . By Theorem 1.1,  $K_{4,2}(2) = 0$ . To determine  $\tilde{H}_4 C_{20}^3$ , note that by Proposition 3.1 and Theorem 1.1,  $K_{5,1}(V, 2)$  contains  $M = S_\lambda(V) \oplus S_\mu(V)$  where  $\lambda = (8, 8, 4)$  and  $\mu = (8, 6, 6)$ . Moreover, using Macaulay2 to compute the dimensions of minimal linear syzygies of the module  $\bigoplus_{k=0}^\infty \text{Sym}^{3k+2}(V)$  with  $\dim V = 4$ , we get  $\dim K_{6,0}(V, 2) = 14003$ . Since  $\bigoplus_{k=0}^\infty \text{Sym}^{3k+2}(V)$  is a Cohen-Macaulay module of codimension 16 with  $h$ -vector  $(10, 16, 1)$ ,

$$\dim K_{5,1}(V, 2) = 14003 - 10 \cdot \binom{16}{6} + 16 \cdot \binom{16}{5} - \binom{16}{4} = 1991.$$

Since  $\dim M = 1991$ ,  $K_{5,1}(V, 2) \cong M$ . By Theorem 1.1,

$$\tilde{H}_4 C_{20}^3 = (8, 8, 4) \oplus (8, 6, 6).$$

Finally,

$$\text{Res } \tilde{H}_5 C_{20}^3 = \tilde{H}_5 C_{19}^3 + \text{Ind } \tilde{H}_4 C_{17}^3 - \tilde{H}_4 C_{19}^3 + \text{Res } \tilde{H}_4 C_{20}^3.$$

This determines  $\tilde{H}_5 C_{20}^3$  as given in our package MatchingComplex.m2.  $\square$

**Proposition 3.4.** *The only non-vanishing homology groups of  $C_{21}^3$  and  $C_{22}^3$  are  $\tilde{H}_5 C_{21}^3, \tilde{H}_5 C_{22}^3, \tilde{H}_6 C_{22}^3$ .*



*Proof.* The computational proof is given in our Macaulay2 package MatchingComplex.m2.  $\square$

**Proposition 3.5.** *The only non-vanishing homology groups of  $C_{23}^3$  are  $\tilde{H}_6 C_{23}^3$  and*

$$\begin{aligned} \tilde{H}_5 C_{23}^3 = & (9, 8, 6) \oplus (8, 6, 6, 3) \oplus (8, 7, 6, 2) \oplus (8, 8, 4, 3) \oplus (8, 8, 5, 2) \\ & \oplus (8, 8, 6, 1) \oplus (9, 6, 6, 2) \oplus (9, 7, 6, 1) \oplus (9, 8, 4, 2) \oplus (9, 8, 5, 1) \\ & \oplus (10, 6, 6, 1) \oplus (10, 8, 4, 1). \end{aligned}$$

*Proof.* Applying the equivariant long exact sequence (1.1) with  $|A| = 23$  we have exact sequences

$$0 \rightarrow \text{Res } \tilde{H}_i C_{23}^3 \rightarrow 0$$

for  $i \neq 5, 6$ , and an exact sequence

$$(3.1) \quad \begin{aligned} 0 \rightarrow \tilde{H}_6 C_{22}^3 \rightarrow \text{Res } \tilde{H}_6 C_{23}^3 \rightarrow \text{Ind } \tilde{H}_5 C_{20}^3 \rightarrow \tilde{H}_5 C_{22}^3 \rightarrow \\ \rightarrow \text{Res } \tilde{H}_5 C_{23}^3 \rightarrow \text{Ind } \tilde{H}_4 C_{20}^3 \rightarrow 0. \end{aligned}$$

Therefore,  $\tilde{H}_i C_{23}^3 = 0$  for  $i \neq 5, 6$ . Moreover, by Proposition 3.3,  $(\text{Ind } \tilde{H}_5 C_{20}^3)_3 = 0$ . Therefore, we have an exact sequence

$$(3.2) \quad 0 \rightarrow (\tilde{H}_5 C_{22}^3)_3 \rightarrow (\text{Res } \tilde{H}_5 C_{23}^3)_3 \rightarrow (\text{Ind } \tilde{H}_4 C_{20}^3)_3 \rightarrow 0.$$

By Proposition 3.1, we know that

$$(\tilde{H}_5 C_{23}^3)_3 = (9, 8, 6).$$

Let  $Y$  be the direct sum of irreducible representations whose corresponding partitions have 4 rows in  $\tilde{H}_5 C_{23}^3$ . Then from exact sequence (3.2), we have

$$\begin{aligned} (\text{Res } Y)_3 &= (\tilde{H}_5 C_{22}^3)_3 + (\text{Ind } \tilde{H}_5 C_{20}^3)_3 - \text{Res}(9, 8, 6) \\ &= (8, 8, 6) \oplus (9, 7, 6) \oplus (9, 8, 5) \oplus (10, 6, 6) \oplus (10, 8, 4). \end{aligned}$$

Therefore,

$$Y_1 = (8, 8, 6, 1) \oplus (9, 7, 6, 1) \oplus (9, 8, 5, 1) \oplus (10, 6, 6, 1) \oplus (10, 8, 4, 1)$$

is a subrepresentation of  $Y$ . By Proposition 3.3,  $\tilde{H}_4 C_{20}^3 = (8, 8, 4) \oplus (8, 6, 6)$ , thus  $\text{Res } Y_1 \oplus \text{Res}(9, 8, 6)$  maps surjectively onto  $\text{Ind } \tilde{H}_4 C_{20}^3$ . Let

$$D = \tilde{H}_5 C_{22}^3 + \text{Ind } \tilde{H}_4 C_{20}^3 - \text{Res } Y_1 - \text{Res}(9, 8, 6).$$

Then the exact sequence (3.1) becomes

$$(3.3) \quad 0 \rightarrow \tilde{H}_6 C_{22}^3 \rightarrow \text{Res } \tilde{H}_6 C_{23}^3 \rightarrow \text{Ind } \tilde{H}_5 C_{20}^3 \rightarrow D \rightarrow \text{Res } Z \rightarrow 0$$

where  $Z = \tilde{H}_5 C_{23}^3 - Y_1 - (9, 8, 6)$ .

By Proposition 3.4,  $\tilde{H}_6 C_{22}^3$  contains only irreducible representations whose corresponding partitions have 8 rows while  $\tilde{H}_5 C_{22}^3$  contains only irreducible representations whose corresponding partitions have at most 6 rows. Thus we have

$$0 \rightarrow \tilde{H}_6 C_{22}^3 \rightarrow (\text{Res } \tilde{H}_6 C_{23}^3)^6 \rightarrow (\text{Ind } \tilde{H}_5 C_{20}^3)^6 \rightarrow 0$$

where  $(\text{Res } \tilde{H}_6 C_{23}^3)^6 = \text{Res } \tilde{H}_6 C_{23}^3 - (\text{Res } \tilde{H}_6 C_{23}^3)_6$  and  $(\text{Ind } \tilde{H}_5 C_{20}^3)^6 = \text{Ind } \tilde{H}_5 C_{20}^3 - (\text{Ind } \tilde{H}_5 C_{20}^3)_6$ . This exact sequence determines the irreducible representations of  $\tilde{H}_6 C_{23}^3$  whose corresponding partitions have 7 or 8 rows as given in our package MatchingComplex.m2. Let

$$C = \text{Ind } \tilde{H}_5 C_{20}^3 + \tilde{H}_6 C_{22}^3 - \text{Res}(\tilde{H}_6 C_{23}^3)^6,$$

where  $(\tilde{H}_6 C_{23}^3)^6$  is the direct sum of irreducible representations in  $\tilde{H}_6 C_{23}^3$  whose corresponding partitions have 7 and 8 rows described above. Then the exact sequence (3.3) becomes

$$(3.4) \quad 0 \rightarrow \text{Res}(\tilde{H}_6 C_{23}^3)_6 \rightarrow C \rightarrow D \rightarrow \text{Res } Z \rightarrow 0.$$

Finally, using Macaulay2 to compute the dimensions of minimal linear syzygies of the module  $\oplus_{k=0}^{\infty} \text{Sym}^{3k+2}(V)$  with  $\dim V = 4$ , we get  $\dim K_{7,0}(V, 2) = 5400$ . Moreover,  $\tilde{H}_4 C_{23}^3 = 0$ , thus by Theorem 1.1,  $K_{5,2}(V, 2) = 0$ . Since  $\oplus_{k=0}^{\infty} \text{Sym}^{3k+2}(V)$  is a Cohen-Macaulay module of codimension 16 with  $h$ -vector  $(10, 16, 1)$ ,

$$\dim K_{6,1}(V, 2) = 10 \cdot \binom{16}{7} - 16 \cdot \binom{16}{6} + \binom{16}{5} - 5400 = 14760.$$

Since the sum of dimensions of irreducible representations  $S_{\lambda}(V)$  corresponding to partitions  $\lambda$  in  $Y_1 \oplus (9, 8, 6)$  is equal to 11520, the sum of dimensions of irreducible representations  $S_{\lambda}(V)$  corresponding to partitions  $\lambda$  in  $Z$  is  $14760 - 11520 = 3240$ . This fact and exact sequence (3.4) determine  $\tilde{H}_5 C_{23}^3$  as stated in the proposition and  $\tilde{H}_6 C_{23}^3$  as given in our package MatchingComplex.m2.  $\square$

**Theorem 3.6.** *The third Veronese embeddings of projective spaces satisfy property  $N_6$ .*

*Proof.* It remains to prove that the only non-zero homology groups of the matching complex  $C_{24}^3$  is  $\tilde{H}_6 C_{24}^3$ . Applying the equivariant long exact sequence (1.1) with  $|A| = 24$  we have exact sequences

$$0 \rightarrow \text{Res } \tilde{H}_i C_{24}^3 \rightarrow 0$$

for  $i \neq 5, 6$ , and an exact sequence

$$0 \rightarrow \tilde{H}_6 C_{23}^3 \rightarrow \text{Res } \tilde{H}_6 C_{24}^3 \rightarrow \text{Ind } \tilde{H}_5 C_{21}^3 \rightarrow \tilde{H}_5 C_{23}^3 \rightarrow \text{Res } \tilde{H}_5 C_{24}^3 \rightarrow 0.$$

Therefore,  $\tilde{H}_i C_{24}^3 = 0$  for  $i \neq 5, 6$  and  $\tilde{H}_5 C_{23}^3$  maps surjectively onto  $\text{Res } \tilde{H}_5 C_{24}^3$ . Moreover, by the result of Ottaviani-Paoletti [OP], the third Veronese embedding of  $\mathbb{P}^3$  satisfies property  $N_6$ . By Theorem 1.1,  $\tilde{H}_5 C_{24}^3$  does not contain any irreducible representations whose corresponding partitions have at most 4 rows. By Proposition 3.5,  $\tilde{H}_5 C_{23}^3$  contains only irreducible representations whose corresponding partitions have at most 4 rows, thus  $\tilde{H}_5 C_{24}^3$  is zero.  $\square$

## APPENDIX

In this appendix we explain the ideas behind our Macaulay2 package MatchingComplex.m2. To determine the homology groups of  $C_N^3$  inductively we use the equivariant long exact sequence (1.1) with  $|A| = N$  to determine two representations  $B$  and  $C$  of  $S_{N-1}$  satisfying the following property.  $B$  is a subrepresentation of  $\text{Res } \tilde{H}_i C_N^3$  and  $C$  is a superrepresentation of  $\text{Res } \tilde{H}_i C_N^3$ . To determine  $\tilde{H}_i C_N^3$ , we need to find a representation  $X$  of  $S_N$  satisfying the property that  $\text{Res } X$  is a subrepresentation of  $C$  and a superrepresentation of  $B$ . The function findEquation in our package will first determine a list  $D$  of all possible partition  $\lambda$  of  $N$  such that  $\text{Res } \lambda$  is a subrepresentation of  $C$ . Let  $X = \sum_{\lambda \in D} x_{\lambda} \cdot \lambda$  where  $x_{\lambda} \geq 0$  is the multiplicity of the partition  $\lambda \in D$  that we need to determine. Restricting  $X$  we have inequalities (obtained by calling findEquation B and findEquation C) expressing the fact that  $\text{Res } \tilde{H}_i C_N^3$  is a subrepresentation of  $C$  and a superrepresentation

of  $B$ . We then use maple to solve for non-negative integer solutions of this system of inequalities.

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